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Small Deviations from Local Equilibrium for a Process Which Exhibits Hydrodynamical Behavior. I

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The symmetric simple exclusion process where infinitely many particles move randomly on $\mathbb{Z}$, jump with equal probability on nearest-neighbor sites, and interact by simple exclusion is considered. It is known that the only extremal invariant measures are Bernoulli, that each measure, in a suitable class, after a "macroscopic" time is locally described, at a zero-order approximation, by a Bernoulli measure with parameter depending on macroscopic space and time, and that the so-defined equilibrium profile satisfies the heat equation. Small deviations from local equilibrium in the hydrodynamical limit are investigated. It is proven, under suitable assumptions, that at first order the state is Gibbs with one- and two-body potentials whose strength depends only on macroscopic space and time and on the equilibrium profile. More precisely, the one-body potential is linear (on the microscopic positions of the particles) and proportional to the macroscopic space gradient of the equilibrium parameter at that time, so that Fourier law holds. The two-body potential varies on a macroscopic scale and does not depend on the microscopic positions of the particles; it is given by the value of the covariance of the Gaussian "macroscopic density fluctuation field."

KEY WORDS: Hydrodynamical behavior of microscopic systems; stochastic dynamics; simple exclusion process; local equilibrium; Fourier law.

1. INTRODUCTION

In this paper we study a stochastic process which describes a system of infinitely many "point hard cores" randomly moving on $\mathbb{Z}$ and interacting by simple exclusion, i.e., the symmetric simple exclusion process. We

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underline some properties exhibited by this process which mimic the hydrodynamical behavior of real physical systems. They might be taken as a first step toward an abstract formulation of "hydrodynamical behavior" and could maybe provide a useful tool in the qualitative analysis of dynamical systems with infinitely many components (particles).

Along these lines many results have recently appeared which concern both the model we consider[1–3] and others.[4–12] The system we study is the simplest among the above ones and so it allows a more detailed analysis; we exploit this by looking at the "small deviations from local equilibrium." We find that they can be described in a very simple way since the states are (locally) Gibbs with space-time dependent potentials representing the deviations from equilibrium; we are indebted to T. Brox and H. Rost for suggesting this line of approach.

In Section 2 we review the features of the model and we underline the analogies with the hydrodynamical properties exhibited by physical systems. We also describe rather informally the results we have obtained, precise statements are given in Section 3, and their proofs can be found in Sections 4 and 5.

2. THE HYDRODYNAMICAL FEATURES OF THE MODEL

The simple exclusion model has been extensively studied both from an "ergodic" point of view (see Ref. 13, for instance) and from a hydrodynamical one (see Refs. 1–3). The system describes infinitely many particles in $\mathbb{Z}$, $\{0,1\}^{\mathbb{Z}}$ is the configuration space, and $\eta(x)$, $x \in \mathbb{Z}$, is the occupation random variable at site x. The time evolution is given by a Markov process as follows: each particle waits, independently of the others, for a Poisson time of mean 1, then decides to jump, at its right or left with probability $\frac{1}{2}$. The jump actually takes place if and only if the chosen site is empty (see Ref. 13 and Section 3 of the present paper for a precise formulation).

P.0. Equilibrium States. The first main feature of the model is that there are infinitely many invariant probability measures, and they form a convex compact (in the weak topology) set with extremal points, $\mathfrak{S}$, which are the Bernoulli measures on $\{0,1\}^{\mathbb{Z}}$ (see Ref. 13). Each element correspond to a point $p \in [0,1]$ via the correspondence

$$\nu_p \in \mathfrak{S}, \qquad \nu_p(\{\eta(x) = 1\}) = p$$

p is the equilibrium parameter; it plays the role that density, energy, and mean velocity have in a real physical system.

P.1. Local Equilibrium. Let μ denote a probability measure on $\{0, 1\}^{\mathbb{Z}}$ and $\mu_t = T_t\mu$ its time evolution according to the simple exclusion

Markov process; then for μ in a suitable class of initial measures

$$\lim d(T_t\mu, \mathcal{E}) = 0 \tag{2.1}$$

where d is a metric on the probability measures given by

$$d(\mu, \nu) = \sup_{x \in \mathbb{Z}} \bar{d}(D_x\mu, D_x\nu)$$

D_x is the space translation as acting on measures, $\bar{d}$ is a distance equivalent to the weak topology, namely,

$$\bar{d}(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n} \| \mu \mid \Lambda_n, \nu \mid \Lambda_n \|$$

where $\| \cdot \|$ is the variational distance, $\Lambda_n = \{-n, \ldots, n\}$ and $\mu \mid \Lambda_n$ is the relativization of μ to the algebra generated by the $\eta(x)$, $x \in \Lambda_n$.

Equation (2.1) states that the system observed in a fixed (finite) region approaches equilibrium. It does not necessarily converge to a point of $\mathcal{E}$ and it might keep wandering closer and closer to $\mathcal{E}$. The equilibrium of another region may initially be different; however, a common behavior is established after some time which depends on the mutual distance between the regions. Only when observations are suitably moved away in space along with time, differences might keep showing up. An equivalent way (more convenient for the sequel) to state Eq. (2.1) is the following: for every (suitable) μ there exists $p(x, t \mid \mu) : \mathbb{Z} \times R_+ \to [0, 1]$ such that for all n and $x_1, \ldots, x_n$ (ν_p below is the Bernoulli measure with parameter p)

$$\lim_{t \to \infty} \sup_{x \in \mathbb{Z}} | \mu_t(\{\eta(x_i + x) = 1, i = 1, \ldots, n\}) - \nu_{p(x,t \mid \mu)}(\{\eta(x_i) = 1, i = 1, \ldots, n\})| = 0 \tag{2.1a}$$

Notice that Eq. (2.1a) does not determine uniquely $p(x, t \mid \mu)$, since if $p'(x, t \mid \mu)$ is such that

$$\lim_{t \to \infty} \sup_{x \in \mathbb{Z}} |p(x, t \mid \mu) - p'(x, t \mid \mu)| = 0$$

then p' fulfills Eq. (2.1a) if p does it.

In the simple exclusion process condition P.1 holds for a class of initial measures whose space correlations decay at infinity.[1,2]

P.2. Hydrodynamical Scaling. The hydrodynamical equations are in a first approximation invariant under suitable space and time rescalings. We want to assume that this is going to hold in some sense also for the microscopic states of the system, states whose local equilibrium structures differ for a space scaling; after that space and a suitable time rescalings should behave the same. This is, however, to be regarded as an "ideal limiting behavior" and as it is for the hydrodynamical equations it is

expected to be valid only in the limit when the gradients become very small. The above argument is mathematically formulated as follows. Let $p(\xi): \mathbb{R} \to [0, 1]$ be a given "smooth" function. For $\epsilon \in (0, 1]$ let μ^ϵ be a probability measure on $\{0, 1\}^{\mathbb{Z}}$ so that for every n, $x_1, \ldots, x_n$, $\xi \in \mathbb{R}$:

$$\lim_{\epsilon \to 0} \left| \mu^\epsilon\left(\left\{\eta\left(x_i + \left[\epsilon^{-1}\xi\right]\right) = 1, i = 1, \ldots, n\right\}\right) - \nu_{p(\xi)}(\{\eta(x_i) = 1, i = 1, \ldots, n\})\right| = 0 \qquad (2.2)$$

$$[a] = \text{integer part of } a$$

The states μ^ϵ have essentially the same local structure when observed at rescaled regions, namely, each μ^ϵ around $\epsilon^{-1}\xi$ is approximately Bernoulli (i.e., in equilibrium) with parameter $p(\xi)$. As ϵ goes to zero the gradients become smaller and the state closer to a real equilibrium.

The hydrodynamical scaling property states that there should exist a function (time scaling)

$$t(\epsilon, \tau) : (0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+, \qquad t(1, \tau) = \tau$$

strictly increasing in τ and decreasing in ϵ such that the following holds. There exists a smooth function $p(\xi, \tau)$ independent of the choice of μ^ϵ but only on $p(\cdot)$, such that for all n, $x_1, \ldots, x_n$, $\xi \in \mathbb{R}$, $\tau \in \mathbb{R}_+$:

$$\lim_{\epsilon \to 0} \left| \mu^\epsilon_{t(\epsilon,\tau)}\left(\left\{\eta\left(x_i + \left[\epsilon^{-1}\xi\right]\right) = 1, i = 1, \ldots, n\right\}\right) - \nu_{p(\xi,\tau)}(\{\eta(x_i) = 1, i = 1, \ldots, n\})\right| = 0 \qquad (2.3)$$

Property P.2 holds for the simple exclusion process (under some further assumptions on the "sequence" μ^ϵ; cf. Section 3) with the choice $t(\epsilon, \tau) = \epsilon^{-2}\tau$ and $p(\xi, \tau)$ satisfies the heat equation with initial condition $p(\xi)$ (see Refs. 1 and 2).

Remarks. For each ϵ the local equilibrium structure of μ^ϵ_t is not unambiguously described by $p(x, t \mid \mu^\epsilon)$; cf. P.1. However, the hydrodynamical limit in Eq. (2.3) removes this ambiguity and determines the limiting hydrodynamical (smooth) profile $p(\xi, \tau)$.

The choice $t(\epsilon, \tau) = \epsilon^{-2}\tau$ is determined by the absence of drift in the simple exclusion process, otherwise a dependence like $\epsilon^{-1}\tau$ would be expected.

Property P.2 provides a derivation of the hydrodynamical equations starting from the microscopic structure of the system.

The variables ξ, τ are usually referred to as "macroscopic space and time," respectively. Reason is that they represent space and time before rescaling, $\epsilon = 1$, and that the rescaling should be considered as a tool to get hydrodynamical behavior in a sharper way. The limit as $\epsilon \to 0$ is called the hydrodynamical limit, and like the thermodynamical limit in statistical

mechanics should be regarded as a mathematically useful idealization of the "real" behavior of the system.

P.3. Stationary Local Equilibria. The typical physical case is that of a system in a bounded region, the walls are in thermal contact with reservoirs, a temperature gradient is established throughout the system, and a stationary heat current flows into the system.

In the simple exclusion model we restrict the available space to the set $-L, \ldots, L$; we modify the process at the sites $\pm L$ by saying that after a "Poisson time" of mean 1 a birth and death process occurs.[1] Namely, with probability p_+ a birth sign occurs at L and it creates a new particle (nothing happens if a particle was already at L). With probability $1-p_+$ a death mark appears and a particle, if present at L, disappears (nothing happens if L was empty). The same occurs at $-L$ with probability p_- and $1-p_-$. The "source" at L forces the system toward equilibrium with parameter p_+, that at $-L$ toward p_-. Let μ_L be the unique invariant measure for this process[1]; then P.3 demands that for all n, $x_1, \ldots, x_n$, $\xi \in (-1, 1)$:

$$\lim_{L\to\infty} \left[\mu_L\left(\{\eta(x_i + [\xi L]) = 1, i = 1, \ldots, n\}\right) - \nu_{p(\xi)}\left(\{\eta(x_i) = 1, i = 1, \ldots, n\}\right)\right] = 0 \tag{2.4}$$

for $p(\xi)$ smooth function. P.3 holds for the simple exclusion model with $p(\xi)$ linear smooth between p_- and p_+, in agreement therefore with P.2 (cf. Ref. 1).

We have so far investigated the macroscopic properties of the system, and, in a way, we have only studied the "thermodynamics" of the model: the aim was always to recognize the equilibrium parameters which better describe in each region the state of the system, the local deviations disappearing after suitable limits: $t \to \infty$ in P.1, $\epsilon \to 0$ in P.2, and $L \to \infty$ in P.3. The next topic to study is the "statistical mechanics," namely, the correlation functions and the local structure of the state: since this is in equilibrium at, let us say, zero order, the hydrodynamically interesting information requires a first order-correction analysis. The aim is to link the local deviations to the macroscopic equilibrium profile via the microscopic structure of the model. A physical example is the Fourier's law: a local observable as the heat flow, describing a local deviation from equilibrium where no steady current is present, is proportional to the temperature gradient, a macroscopic quantity, via the conductivity coefficient, i.e., one of the characteristic features of the system.

The purpose of this paper is to offer a contribution along this line; the results until new material is available (to be hopefully provided by the analysis of other models) and their interpretation are therefore at a rather

provisional stage. Our presentation will be rather informal and qualitative in this section; precise statements are given in the next one.

We start considering the time-dependent case; cf. P.2. A possible choice for the initial measure μ^ϵ is a Gibbs state with fixed translationally invariant two-body interaction (exponentially decaying) whose strength is given by a factor which vanishes as ϵ goes to zero: it is important to notice that any speed for such convergence is allowed. The one-body term is adjusted so that Eq. (2.2) holds, i.e., as ϵ vanishes μ^ϵ determines an equilibrium profile $\rho(\xi)$. More general cases are actually treated (see Section 3) but we think that the above keeps the main features of what we want to describe.

After a time $\epsilon^{-2}\tau$, $\tau > 0$, the state is again locally Gibbs. The Hamiltonian is purely one body if terms of order ϵ are neglected and local equilibrium is reached, as was described in P.2. Deviations from this behavior appear at order ϵ. For small values of τ they are described by a linear microscopic space-dependent one-body potential. Even though the relevant contribution to the correction at time $\tau = 0$ might arise from the correlations due to the two-body interaction, they completely disappear in a microscopic time, i.e., for any $\tau > 0$. The ϵ-small space-dependent one-body potential is responsible for a non-zero average current flowing in the system, which is stationary with respect to microscopic time since the state changes only in the macroscopic time scale of τ. The present picture explains in which sense flows could occur in a system whose stationary states do not have nonzero average currents.

The value of the average current is proportional to the intensity of the ϵ-dependent one-body potential, i.e., to the equilibrium profile gradient at that space and time, so that Fourier's law is satisfied in this model. We remark that the system is not macroscopically stationary; however, Fourier's law holds at each time. Even though usually stated for stationary cases, it is often used in the derivation of hydrodynamical equations (as for the heat equation) also in nonstationary situations, and this model provides a microscopic justification for this procedure.

The origin of the one-body potential is quite clear. The initial correlations in the states $\mu_{\epsilon^{-2}\tau}^\epsilon$ are disappearing; therefore we define

$$\pi(x,t;\epsilon) = \mu_t^\epsilon(\{\eta(x) = 1\}) \tag{2.5}$$

Then a good approximation for μ_t^ϵ at $t = \epsilon^{-2}\tau$ and τ small is expected to be

$$\mu_\tau^\epsilon \sim \exp\left\{ \sum_x \log \frac{\pi(x,t;\epsilon)}{1-\pi(x,t;\epsilon)} \eta(x) + \log[1-\pi(x,t;\epsilon)] \right\} \tag{2.6}$$

The right-hand side of Eq. (2.6) is a formal writing for the product of independent measures for each $\eta(x)$, with averages given by Eq. (2.5). By limiting the sum to x in Λ (bounded) Eq. (2.6) is well defined: we

approximate $\pi(x + [\epsilon^{-1}\xi], \epsilon^{-2}\tau; \epsilon)$ with $p(\xi + \epsilon x, \tau)$ in agreement with P.2; then we expand $p(\xi + \epsilon x, \tau)$ around $p(\xi, \tau)$ and we derive from Eq. (2.6) the one-body potential.

As the macroscopic time τ increases, correlations build up in the system and a two-body potential adds up to the previous one. This new Gibbs factor for a region Λ fixed around $[\epsilon^{-1}\xi]$ reads as

$$\exp\left\{ \epsilon\gamma_\tau \frac{1}{2} \sum_{\substack{x \neq y \\ x,y \in \Lambda}} [\eta(x) - p][\eta(y) - p] \right\}, \qquad p = p(\xi, \tau) \tag{2.7}$$

Notice that up to order ϵ this factor does not change the average value of $\eta(x)$ and that compatibility holds (up to order ϵ), namely, that the state for a region $\Lambda' \supset \Lambda$ agrees on $\{\eta(x), x \in \Lambda\}$ with the state for the region Λ. γ_τ in Eq. (2.7) depends on ξ and τ, as we will see in more detail below. The interaction given by Eq. (2.7) is of a macroscopic nature, since it does not decay on a microscopic scale, and Λ is fixed as ϵ goes to zero.

As τ increases the above hydrodynamical approximation fails and $\pi(x, \epsilon^{-2}\tau; \epsilon)$ is not anymore close to $p(\epsilon x, \tau)$. However, Eq. (2.6) holds and gives a uniformly good approximation for all t as ϵ goes to zero (cf. Note 1 at end of paper) and the assumptions we made on the sequence μ^ϵ. It may also be easily seen that

$$|\pi(x, t; \epsilon) - \pi(y, t; \epsilon)| \lesssim t^{-1/2}|x - y|$$

and therefore an approximate local equilibrium structure holds at all times and becomes uniformly more accurate as ϵ goes to zero.

In the stationary case, P.3, the same picture holds. As L diverges a linear local equilibrium profile appears. Its local deviations are of order L^{-1} and are described by one- and two-body potentials. The former is linear in space, the same as the one in the time-dependent case; the two-body term also looks like the one in the time-dependent case, its strength being determined by the equilibrium profile, i.e., the values of the parameters at the boundaries (cf. Note 2).

We conclude this section with some remarks which show the connection between our approach and the scheme developed in Ref. 3 and which somehow clarifies the origin of the two-body potential we have been discussing so far. The aim in Ref. 3 is to introduce observables, hereafter called fields, whose values reveal the equilibrium profile structure exhibited by the system. The natural candidate, of course, is the particle density in a large region, whose distribution values at equilibrium become peaked around the equilibrium parameter as the size of the region increases (law of large numbers). For this reason one introduces the density field

$$p^\epsilon(f, \tau) = \epsilon \sum_{x \in \mathbb{Z}} f(\epsilon x)\eta(x, \epsilon^{-2}\tau) \tag{2.8}$$

for $f \in \mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing C^∞ functions. The analog of P.2 is that the so-defined field converges as ϵ goes to zero to a deterministic field, namely, for all $\delta > 0$:

$$\lim_{\epsilon \to 0} \mu^\epsilon\left(\left\{\left|p^\epsilon(f,\tau) - \int d\xi\, f(\xi) p(\xi,\tau)\right| < \delta\right\}\right) = 1 \tag{2.9}$$

where $p(\xi,\tau)$ is the same function as that defined in P.2 (Ref. 3). Notice that Eq. (2.9) can be derived independently of the previous considerations and $p(\xi,\tau)$ and the hydrodynamical equations obtained without going through P.2: in that case $p(\xi,\tau)$ cannot be interpreted as a local equilibrium parameter and it happens that no local equilibrium structure holds even though Eq. (2.9) is satisfied.[12] Quite naturally the next object to study is the density fluctuation field, namely,

$$\mathcal{F}^\epsilon(f,\tau) = \sqrt{\epsilon} \sum_{x \in \mathbb{Z}} f(\epsilon x)\left[\eta(x,\epsilon^{-2}\tau) - \mathbf{E}_{\mu^\epsilon}\left(\eta(x,\epsilon^{-2}\tau)\right)\right] \tag{2.10}$$

This has been done in Ref. 3 (their results have been obtained before this paper was written and we are deeply indebted to the authors for keeping us informed on progress along this line as it appeared) where it was proven that as ϵ goes to zero the fluctuation field becomes Gaussian. Its covariance can be written as

$$\int d\xi\, d\eta\, \tilde{C}_\tau(\xi,\eta) f(\xi) f(\eta) + \int d\xi\, f(\xi)^2\left[p(\xi,\tau) - p^2(\xi,\tau)\right] \tag{2.11}$$

where $\tilde{C}_\tau$ is an explictly given smooth function.

We now come back to our approximation scheme. We need a stronger version; we prove in fact that there exists a function $\gamma_\tau(\xi,\eta)$ such that for each fixed n

$$\lim_{\epsilon \to 0} \sup_{x_1,\ldots,x_n} \epsilon^{-1} \Bigg[\mu^\epsilon_{\epsilon^{-2}\tau}\left(\{\eta(x_i) = 1,\ i = 1,\ldots,n\}\right) - \exp\Bigg\{ \sum_{i=1}^n \Bigg[\log \frac{\pi(x_i,\epsilon^{-2}\tau;\epsilon)}{1 - \pi(x_i,\epsilon^{-2}\tau;\epsilon)}\, \eta(x_i) + \log\left(1 - \pi(x_i,\epsilon^{-2}\tau;\epsilon)\right) \Bigg] + \epsilon \frac{1}{2} \sum_{i \neq j} \gamma_\tau(\epsilon x_i, \epsilon x_j)\left(\eta(x_i) - \pi(x_i,\epsilon^{-2}\tau;\epsilon)\right) \times \left(\eta(x_j) - \pi(x_j,\epsilon^{-2}\tau;\epsilon)\right) \Bigg\} \Bigg] = 0 \tag{2.12}$$

Choosing $x_i = [\epsilon^{-1}\xi] + x_i'$ with x_i' and ξ fixed we recover the previous case, and so γ_τ in Eq. (2.7) is equal to $\gamma_\tau(\xi,\xi)$. On the other hand, by Eq. (2.12) we can easily compute the covariance of $\mathcal{F}^\epsilon(f,\tau)$ and we find in the limit ϵ going to zero that $\bar{\gamma}_\tau(\xi,\eta) = \tilde{C}_\tau(\xi,\eta)$. In particular, therefore, the value of the interaction strength $\bar{\gamma}_\tau$ is the value on the diagonal of the smooth part, $\tilde{C}_\tau(\xi,\xi)$, of the covariance of the macroscopic density fluctuation field.

A final remark: the covariance $\tilde{C}_\tau(\xi,\eta)$ obeys a "natural" equation given by the martingale condition for the fluctuation field and can therefore be computed directly without going through the above approximation scheme.

3. RESULTS

The simple exclusion process[13] is the Markov process with state space $\{0,1\}^{\mathbb{Z}}$ and generator L which acts on the cylindrical functions f as

$$(LF)(\eta) = (1/2)\sum_{x\in\mathbb{Z}}\left[f(\eta(x,x+1)) - f(\eta)\right]$$
$$(\eta(x,x+1))(x) = \eta(x+1), \qquad (\eta(x,x+1))(x+1) = \eta(x) \tag{3.1}$$
$$(\eta(x,x+1))(y) = \eta(y), \qquad y \neq x, x+1$$

Definition 3.1: Local Equilibrium. We say that μ^ϵ, $\epsilon \in (0,1]$ defines a local equilibrium structure with parameter $p(\xi)$ if the following conditions hold:

H.1. $p(\xi):\mathbb{R}\to[0,1]$ is C^3 and all derivatives up to third order are uniformly bounded.

H.2. $\lim_{\epsilon\to 0}\sup_{x\in\mathbb{Z}}\epsilon^{-1}|\mu^\epsilon(\{\eta(x)=1\}) - p(\epsilon x)| = 0$.

H.3. There are functions $\varphi_n:\mathbb{N}\to\mathbb{R}_+$, $n=2,3,\ldots,$ such that (i) $\sum_{x\in\mathbb{Z}}\varphi_n(|x|) < +\infty$ and (ii) for all positive integers k_1, k_2 and $x_{k_1} < \cdots < x_1 < y_1 < \cdots < y_{k_2}$ the following holds:

$$\begin{aligned}&|\mu^\epsilon(\{\eta(x_i)=1, \eta(y_j)=1, i=1,\ldots,k_1, j=1,\ldots,k_2\})\\ &\quad - \mu^\epsilon(\{\eta(x_i)=1, i=1,\ldots,k_1\})\\ &\quad \times \mu^\epsilon(\{\eta(y_j)=1, j=1,\ldots,k_2\})| \leqslant \varphi_{k_1+k_2}(|x_1-y_1|), \qquad \forall\epsilon\in(0,1]\end{aligned}$$

H.4. For every $k\in\mathbb{N}$, $x_1,\ldots,x_k$, the following holds:

$$\lim_{\epsilon\to 0}\sup_{x\in\mathbb{Z}}|\mu^\epsilon(\{\eta(x_i+x)=1, i=1,\ldots,k\}) - p^k(\epsilon x)| = 0$$

Theorem 3.1. Assume that μ^ϵ defines a local equilibrium structure with parameter $p(\xi)$; see Definition 3.1. Let $p(\xi,\tau)$ be the solution of

$$\frac{\partial}{\partial t}p(\xi,\tau) = \frac{1}{2}\frac{\partial^2}{\partial\xi^2}p(\xi,\tau), \qquad p(\xi,0) = p(\xi)$$

For every $\xi \in \mathbb{R}$, $\tau > 0$, Λ bounded, and $\eta_\Lambda = \{\eta_x, x \in \Lambda\} \in \{0,1\}^{|\Lambda|}$, define for $p(\xi,\tau) \neq 0, 1$

$$\mathfrak{A}_{\xi,\tau,\Lambda}(\eta_\Lambda) = \sum_{x\in\Lambda} \alpha x(\eta_x - p) + \frac{1}{2} \sum_{\substack{x\neq y \\ x,y\in\Lambda}} \gamma_\tau(\eta_x - p)(\eta_y - p) \tag{3.2}$$

$$\alpha = \frac{1}{p(1-p)}\, p', \qquad p = p(\xi,\tau), p' = \frac{\partial}{\partial\xi}\, p(\xi,\tau) \tag{3.2a}$$

$$\begin{gathered} \gamma_\tau = \gamma_\tau(\xi,\xi), \\ \bar\gamma_\tau(\xi,\eta) = \frac{1}{p(\xi,\tau)\big[1-p(\xi,\tau)\big]}\, \frac{1}{p(\eta,\tau)\big[1-p(\eta,\tau)\big]}\, \bar\gamma_\tau(\xi,\eta) \end{gathered} \tag{3.2b}$$

$$\begin{aligned} \xi \leqslant \eta : \bar\gamma_\tau(\xi,\eta) = {} & \int dx\, dy\, \pi_\tau(x,y\,|\,\xi,\eta)\big[p'(x)p(y) - p'(y)p(x)\big]1(x>y) \\ & - \int_0^\tau d\sigma \int dz_1\, dz_2\, du\, \pi_\sigma(u,u\,|\,\xi,\eta) \\ & \times \pi_{\tau-\sigma}(z_1,z_2\,|\,u,u)p''(z_1)p(z_2) \end{aligned} \tag{3.2c}$$

$$\pi_\sigma(u,v\,|\,\xi,\eta) = \frac{1}{2\pi\sigma} \exp\Big\{ -\frac{1}{2\sigma}\big[(u-\xi)^2 + (v-\eta)^2\big]\Big\} \tag{3.2d}$$

Then

$$\begin{aligned} \lim_{\epsilon\to 0} \epsilon^{-1}\Big[& \mu^\epsilon_{\epsilon^{-2}\tau}\Big(\big\{\eta\big(x + [\epsilon^{-1}\xi]\big) = \eta_x\,,\ \forall x \in \Lambda\big\}\Big) \\ & - \nu_{p(\xi_\epsilon,\tau)}\big(\{\eta(x) = \eta_x\,,\ \forall x \in \Lambda\}\big)\exp\big(\epsilon\mathfrak{A}_{\xi,\tau,\Lambda}(\eta_\Lambda)\big)\Big] = 0 \end{aligned} \tag{3.3}$$

where $\xi_\epsilon = \epsilon[\epsilon^{-1}\xi]$. Furthermore, for each n Eq. (2.12) holds. Notice that the restriction $p(\xi,\tau) \neq 0, 1$ can be left out by assuming that $p(\xi,0)$ is not identically 0 or 1.

Theorem 3.1 will be proven in Section 5.

As already remarked in Section 2, Eq. (2.12) was obtained when $n = 2$ in Ref. 3 and $\bar\gamma_\tau(\xi,\eta)$ is the smooth part of the covariance $\tilde C(\xi,\eta)$ [see Eq. (2.11)] of the macroscopic limiting fluctuation field.

4. THE COUPLING TECHNIQUE

To prove the theorems of Section 3 we need good estimates on the simple exclusion process for finitely many particles (because the process is additive[13]). We compare it with the free one by introducing couplings between the two processes, different ones according to the property to be proven. They are all modifications of that introduced in Ref. 2 and are grouped together in this section.

We set n as the number of particles, $\mathbf{x}(t) = x_1(t), \ldots, x_n(t)$ $[\mathbf{x}^0(t) = x_1^0(t), \ldots, x_n^0(t)]$ the position random variables in the interacting (free) process and $\mathbf{x}(0) = \mathbf{x}^0(0) = \mathbf{x}: P_x[P_{\mathbf{x}}^0]$ their law. The main coupling process is Q and is defined by its generator $\mathfrak{L}$, which acts on functions $f(\mathbf{x}, \mathbf{x}^0)$ as

$$(\mathfrak{L}f)(\mathbf{x}, \mathbf{x}^0) = \sum_{i=1}^{n} \sum_{z_i = \pm 1} \frac{1}{2} \left\{ f\big(\mathbf{x}(x_i, x_i + z_i), (x_1^0, \ldots, x_i^0 + z_i, \ldots, x_n^0)\big) - f(\mathbf{x}, \mathbf{x}^0) \right\} \tag{4.1}$$

$$\mathbf{x}(x_i, y) = \begin{cases} x_1 \ldots x_{i-1} y \ldots x_n & \text{if} \quad y \notin \{x_1 \ldots x_n\} \\ \mathbf{x} & \text{if} \quad y \in \{x_1 \ldots x_{i-1}\} \\ x_1 \ldots x_{j-1} y \ldots x_{j-1} x_i \ldots x_n & \text{if} \quad y = x_j, j > i \end{cases} \tag{4.2}$$

The process Q starts at $(\mathbf{x}, \mathbf{x})$. The following properties hold[2,14]:

R.1. The process Q is a coupling, i.e., its relativitation to the σ algebra generated by the paths of the interacting (free) particles is the simple exclusion (independent) process for those particles. It defines a homomorphic mapping from the free onto the interacting trajectories; see also R.8 below.

R.2. Free and interacting particles have the same displacements, unless this violates the exclusion condition: in that case the rule is given by Eq. (4.2). According to it we can write

$$x_i(t) - x_i^0(t) = \sum_{j<i} D_{i,j}(t) \tag{4.3}$$

where $D_{i,j}(t)$ gives the contribution due to displacements of $x_i^0(t') - x_j^0(t')$ which would have taken $x_i(t')$ and $x_j(t')$ onto each other; here $0 < t' \leqslant t$.

R.3. The Q process relativized to the σ algebra generated by the first k free and interacting particles is the same as the process defined for only those k particles, disregarding the others.

R.4. $x_1(t) = x_1^0(t)$ at all times.

R.5. The defintion of Q is asymmetric on the labels of the particles: the smaller are matched "better" than the others. To compare paths of given labeled particles it is therefore convenient to introduce couplings Q^π, where π is a permutation $\pi(1) \ldots \pi(n)$ of $1, \ldots, n$. In Q^π the role of particle j is played by $l: \pi(l) = j$ so it is particle $i: \pi(i) = 1$ in Q^π which moves the same in the two processes.

We will need estimates on $|x_i(t) - x_j(t)|$ in terms of $|x_i^0(t) - x_j^0(t)|$. It is

convenient to introduce the coupling $\tilde{Q}$ whose generator $\tilde{\mathcal{L}}$ is

$$(\tilde{\mathcal{L}}f)(\mathbf{x},\mathbf{x}^0)=\sum_{i=1}^{n}\sum_{z_i=\pm 1}\tfrac{1}{2}\Big\{f\big((\mathbf{x})(x_i,\tilde{x}_1+z_i),\big(x_1^0,\ldots,x_i^0+z_i,\ldots,x_n^0\big)\big)$$

$$-f(\mathbf{x},\mathbf{x}^0)\Big\} \tag{4.4}$$

where

$$\mathbf{x}(x_i,\tilde{x}_i+z_i)=(x_2,x_1,x_3,\ldots,x_n)$$

$$\text{if}\quad x_i+z_i=x_j,\quad \{i=1,j=2\}\cup\{i=2,j=1\} \tag{4.5a}$$

$$\mathbf{x}(x_i,\tilde{x}_i+z_i)=\mathbf{x}(x_i,x_i+z_i)\qquad\text{in the other cases} \tag{4.5b}$$

R.6. The mapping defined by $\tilde{Q}$ is such that (i) $x_1<x_2$ if $x_1^0\leqslant x_2^0$ and $x_1>x_2$ if $x_1^0\geqslant x_2^0$, (ii) $0\leqslant|x_1(t)-x_2(t)|-|x_1^0(t)-x_2^0(t)|\leqslant 1$. Property R.3 for $k\geqslant 2$ holds also for $\tilde{Q}$.

R.7. The main probability estimate. For every $\alpha>1/4$ there are A' and $A>0$ so that

$$Q\Big(\big\{|x_i(t)-x_i^0(t)|<t^\alpha,\ i=1,\ldots,n,\ \forall t\geqslant T\big\}\Big)>1-A'\exp(-AT^{\alpha-1/4}) \tag{3.6}$$

Equation (4.6) holds also for Q^π, $\tilde{Q}$, $\tilde{Q}^\pi$.

For notational convenience we define "marks" as random variables in the coupled process:

R.8. For $x\in\mathbb{Z}$, $t\in\mathbb{R}_+$, $M_{x,x+1}(t)$ is defined as

$$M_{x,x+1}(t)=1\qquad\text{if }\exists i,j: x_i(t)=x, x_j(t)=x+1,\ \lim_{t'\uparrow t}x_i(t')=x+1$$

$$M_{x,x+1}(t)=-1\quad\text{if }\exists i,j:\ \lim_{t'\uparrow t}x_i(t')=x_i(t)=x,\ \lim_{t'\uparrow t}x_j(t')=x_j(t)=x+1,$$

$$\lim_{t'\uparrow t}\big[x_i^0(t'),x_j^0(t')\big]\neq\big[x_i^0(t),x_j^0(t)\big]$$

$$M_{x,x+1}(t)=0\qquad\text{otherwise}$$

The random variables $M_{x,x+1}(t)$, $x\in\mathbb{Z}$, $t\in\mathbb{R}_+$ define the so-called "marks process." It is easy to see that any of the couplings Q^π and $\tilde{Q}^\pi$ induces the same law on the simple exclusion "marked" process, hereafter called the completed interacting process, no reference to the free one being needed: the couplings then define isomorphic mappings between the free and the completed interacting process. We will call the trajectories in the latter complete trajectories.

R.9. Like the simple exclusion also the completed interacting process is additive. Namely, if $\mathcal{T} \subset \{1, \ldots, n\}$ the process of the particles in $\mathcal{T}$ and of those marks which only involve particles in $\mathcal{T}$ is just the same as only the $\mathcal{T}$ particles were present.

5. PROOF OF THEOREM 3.1

We first prove that the initial correlations decay in a microscopic time, namely, let $\bar{\mu}^\epsilon$ be the Bernoulli measure on $\{0,1\}^{\mathbb{Z}}$ such that

$$\bar{\mu}^\epsilon(\{\eta(x) = 1\}) = \mu^\epsilon(\{\eta(x) = 1\}), \qquad \forall x \in \mathbb{Z} \tag{5.1}$$

Then we have the following:

Theorem 5.1. Let the assumptions H.3 and H.4 of Section 3 hold. Then for every $n, x_1 \ldots x_n, \tau > 0$, the following holds:

$$\lim_{\epsilon \to 0} \epsilon^{-1} \sup_{x \in \mathbb{Z}} |\mu^\epsilon_{\epsilon^{-2}\tau}(\{\eta(x_i + x) = 1, i = 1, \ldots, n\}) - \bar{\mu}^\epsilon_{\epsilon^{-2}\tau}(\{\eta(x_i + x) = 1, i = 1, \ldots, n\})| = 0 \tag{5.2}$$

Proof. For notational simplicity we pose $x = 0$, the estimate being uniform in $x \in \mathbb{Z}$. Let $P_{\mathbf{x}}$ be the law of the simple exclusion process for particles starting at $\mathbf{x} = x_1 \ldots x_n$; then by duality Eq. (5.2) is reduced to the proof that

$$\lim_{\epsilon \to 0} V(\epsilon, \mathbb{Z}^n) = 0 \tag{5.3a}$$

$$A \subseteq \mathbb{Z}^n : V(\epsilon, A) = \epsilon^{-1} \sum_{\mathbf{z} \in A} P_{\mathbf{x}}(\{\mathbf{x}(t) = \mathbf{z}\}) \times \Big[\mu^\epsilon(\{\eta(z_i) = 1, i = 1, \ldots, n\}) - \prod_{i=1}^{n} \mu^\epsilon(\{\eta(z_i) = 1, i = 1, \ldots, n\}) \Big] \tag{5.3b}$$

$$t = \epsilon^{-2}\tau \tag{5.3c}$$

For $i_1 \neq j_1$, $i_2 \neq j_2$, $\mathrm{Card}(i_1, j_1, i_2, j_2) \geqslant 3$ and $i_1 j_1 i_2 j_2$ in the set $\{1 \ldots n\}$ for $\frac{1}{2} > \alpha > \frac{1}{4}$ define

$$A(i_1 j_1 i_2 j_2) = \{\mathbf{z} \in \mathbb{Z}^n : |z_{i_1} - z_{j_1}| \leqslant |z_{i_2} - z_{j_2}| \leqslant |z_k - z_{k'}|, \forall k \neq k', (k, k') \neq (i_1, j_1) \text{ and } |z_{i_1} - z_{j_1}| := l_1 \leqslant t^\alpha, |z_{i_2} - z_{j_2}| : l_2 \leqslant t^\alpha\}$$

Writing in the following equation A for $A(i_1 j_1 i_2 j_2)$ we have for c and c'

large enough

$$
\begin{aligned}
|V(\epsilon, A)| &\leqslant \epsilon^{-1} c \sum_{\mathbf{z} \in A} P_{\mathbf{x}}(\{\mathbf{x}(t) = \mathbf{z}\}) \varphi_n(l_1) \\
&\leqslant \epsilon^{-1} c \sum_{l_1 \leqslant t^\alpha} \sum_{l_2 \leqslant t^\alpha} P_{\mathbf{x}}(\{|x_{i_1}(t) - x_{j_1}(t)| = l_1, |x_{i_2}(t) - x_{j_2}(t)| = l_2\}) \varphi_n(l_1) \\
&\leqslant \epsilon^{-1} c' \sum_{l_1 \leqslant 2t^\alpha} \sum_{l_2 \leqslant 2t^\alpha} \left(\frac{\epsilon}{\sqrt{\tau}} \right)^2 \varphi_n(l_1) + \epsilon^{-1} c' A' \exp(-A t^{\alpha - 1/4}) \\
&\leqslant \epsilon^{1-2\alpha} c' 3\tau^{-1+\alpha} \sum_{x \geqslant 1} \varphi_n(x) + \epsilon^{-1} c' A' \exp(-A \tau^{\alpha - 1/4} \epsilon^{-2(\alpha - 1/4)}) \qquad (5.4)
\end{aligned}
$$

where H.3 and the definition of A have been used to get the first inequality; the third one is derived using the coupling $\tilde{Q}^\pi$ with $\pi(i_1) = 1$, $\pi(j_1) = 2$ (see Section 4, R.6) and Eq. (4.6) and well-known estimates for free particles.

Let $B(1,2)$ be the set in $\mathbb{Z}^n$ disjoint from all $A(i_1, j_1, i_2, j_2)$'s and such that $|z_i - z_j| \geqslant |z_1 - z_2|$ for all $i \neq j$. For every $\gamma > 0$ define $L(\gamma)$ so that

$$
\sum_{x \geqslant L(\gamma)} \varphi_n(x) < \gamma \qquad (5.5)
$$

and let ϵ be such that $t^\alpha > L(\gamma)$. Then for c, c' large enough

$$
\begin{aligned}
&|V(\epsilon, B(1,2))| \\
&\quad \leqslant \epsilon^{-1} c \sum_{l \geqslant L(\gamma)} \sum_{\substack{\mathbf{z} \in B(1,2) \\ |z_1 - z_2| = l}} P_{\mathbf{x}}(\{\mathbf{x}(t) = \mathbf{z}\}) \varphi_n(l) \\
&\qquad + \epsilon^{-1} c \sum_{l < L(\gamma)} \sum_{\substack{\mathbf{z} \in B(1,2) \\ |z_1 - z_2| = l}} P_{\mathbf{x}}(\{\mathbf{x}(t) = \mathbf{z}\}) \\
&\qquad \times \left\{ \varphi_n(t^\alpha)' + \prod_{i>2} \mu^\epsilon(\{\eta(z_i) = 1\}) \right. \\
&\qquad\quad \left. \times \left| \mu^\epsilon(\{\eta(z_i) = 1, i = 1,2\}) - \prod_{i=1}^{2} \mu^\epsilon(\{\eta(z_i) = 1\}) \right| \right\} \\
&\quad \leqslant \epsilon^{-1} c' \frac{\epsilon}{\sqrt{\tau}} \gamma + \epsilon^{-1} L(\gamma) c' \frac{\epsilon}{\sqrt{\tau}} \\
&\qquad \times \left\{ \varphi_n(\epsilon^{-2\alpha} \tau^\alpha) + \sup_{\substack{z_1 \neq z_2 \\ |z_1 - z_2| \leqslant L(\gamma)}} \left| \mu^\epsilon(\{\eta(z_i) = 1, i = 1,2\}) \right. \right. \\
&\qquad\qquad \left. \left. - \prod_{i=1}^{2} \mu^\epsilon(\{\eta(z_i) = 1\}) \right| \right\} \qquad (5.6)
\end{aligned}
$$

where the coupling $\tilde{Q}$ (see Section 4, R.6) has been used. By assumption H.4 the last term in curly brackets vanishes as $\epsilon \to 0$ so that the contribution of $V(\epsilon, B(1,2))$ is also arbitrary small. For $B(i,j)$ we use the coupling $\tilde{Q}^{\pi}$, $\pi(i) = 1$, $\pi(j) = 2$ and the same estimate as in Eq. (5.6) is obtained. The theorem is therefore proven. ■

The contribution which gives rise to the one-body potential is estimated in the following:

Theorem 5.2. Let the assumptions H.1–H.4 hold. Define ξ_ϵ as

$$\xi_\epsilon = \epsilon\left[\epsilon^{-1}\xi\right]$$

Then for every n, $x_1, \ldots, x_n$

$$\lim_{\epsilon\to 0} \epsilon^{-1}\left\{\left[\mu_{\epsilon^{-2}\tau}^{\epsilon}\left(\left\{\eta\left(x_i + \left[\epsilon^{-1}\xi\right]\right) = 1, i = 1, \ldots, n\right\}\right) - p(\xi_\epsilon, \tau)^n\right]\right.$$

$$\left. - \left[\sum_{\mathbf{z}}\left(P\left(\left\{\mathbf{x}(t) = \mathbf{z}\right\}\right) - P^0\left(\left\{\mathbf{x}^0(t) = \mathbf{z}\right\}\right)\right)\prod_{i=1}^{n} p(\epsilon z_i)\right]\right\}$$

$$= p(\xi,\tau)^{n-1} p'(\xi,\tau)\sum_{i=1}^{n} x_i \tag{5.7}$$

where P, $[P^0]$ is the law of n interacting (free) particles starting at $x_1 + [\epsilon^{-1}\xi], \ldots, x_n + [\epsilon^{-1}\xi]$, $p'(\xi,\tau) = (\partial/\partial\xi)p(\xi,\tau)$, and

$$p(\xi,\tau) = \lim_{\epsilon\to 0}\sum_{z} P\left(\left\{x(\epsilon^{-2}\tau) = z\right\}\right)p(\epsilon z) = \left(\frac{1}{2\pi\tau}\right)^{1/2} dx\, e^{-x^2/2\tau} p(x + \xi) \tag{5.8}$$

where P is the law of a single random walk starting at $[\epsilon^{-1}\xi]$.

Proof. We use duality and Theorem 5.1. ■

Lemma 4.1. We have that

$$\lim_{\epsilon\to 0}\epsilon^{-1}\sum_{\mathbf{z}}\left[P\left(\left\{\mathbf{x}(t) = \mathbf{z}\right\}\right) - P^0\left(\left\{\mathbf{x}^0(t) = \mathbf{z}\right\}\right)\right]\prod_{i=1}^{n} p(\epsilon z_i)$$

$$= \lim_{\epsilon\to 0}\epsilon^{-1}\sum_{\mathbf{z},\mathbf{z}'} Q\left(\left\{\mathbf{x}(t) = \mathbf{z}\right\}\left\{\mathbf{x}(t) = \mathbf{z}'\right\}\right)$$

$$\times \sum_{\{n_i\}}{}' \prod_{i=1}^{n}\left[p^{(n_i)}(\epsilon z_i)\left(\frac{1}{2}\right)^{n_i - 1}\left(\epsilon(z_i - z_i')^{n_i}\right)\right] \tag{5.9}$$

where Q is defined in R.1, section 4, $n_i \in \{0,1,2\}$, $\sum'$ is the sum extended to those $\{n_i\}$ such that $0 < \sum_{i=1}^{n} n_i \leqslant 2$, $p^{(n_i)}(\cdot)$ is the n_ith derivative of $p(\cdot)$.

Proof. Since Q is a coupling we can write

$$\lim_{\epsilon\to 0}\epsilon^{-1}\sum_{\mathbf{z}}\Big[P\big(\{\mathbf{x}(t)=\mathbf{z}\}\big)-P^0\big(\{\mathbf{x}^0(t)=\mathbf{z}\}\big)\Big]\prod_{i=1}^{n}p(\epsilon z_i)$$

$$=\lim_{\epsilon\to 0}\epsilon^{-1}\sum_{\mathbf{z},\mathbf{z}'}Q\big(\{\mathbf{x}(t)=\mathbf{z}\}\{\mathbf{x}^0(t)=\mathbf{z}'\}\big)$$

$$\times\Big(\prod_{i=1}^{n}p(\epsilon z_i)-\prod_{i=1}^{n}p(\epsilon z_i')\Big)$$

By R.7, Eq. (4.6), with $\frac{1}{4}<\alpha<\frac{1}{3}$ we can disregard those $\mathbf{z},\mathbf{z}'$ such that $|z_i-z_i'|>t^\alpha$ for some i. We now expand in $\epsilon(z_i'-z_i)$ and the third-order contribution behaves as

$$\epsilon^{-1}\big((\epsilon^{-2}\tau)^{\alpha}\epsilon\big)^3$$

which is vanishing as ϵ goes to zero, because $\alpha<\frac{1}{3}$. ■

Definition 5.1. Given $\mathcal{T}\subset\{1,\ldots,n\}$ and $t>0$ we define the stopping time T_1 on the completed interacting process (cf. R.8) as the first time before t when a nonzero mark occurs which refers to a pair of particles both in $\mathcal{T}$; otherwise we put $T_1=\infty$. $T_2\geqslant T_1$, $T_3\geqslant T_2$, and so on, are defined analogously. Given a complete trajectory with $T_m<\infty$ we construct a new trajectory by changing the value of the mark at T_m and by interchanging the role of the two particles (specified by the mark at T_m) at times after T_m. Notice that the stopping times $(T_n)_{n\in\mathbb{N}}$ have the same values in the new trajectory. By iterating this procedure we define a measurable partition whose atoms are denoted by g_t. $d\overline{P}(g_t)$ is their probability distribution. g_t fixes the trajectories of the particles not in $\mathcal{T}$ and up to permutation those in $\mathcal{T}$. More precisely a trajectory belonging to g_t is specified once the ordering at all times ($\leqslant t$) of the $\mathcal{T}$ particles is given. The ordering might change only at the times T_m with $T_m<\infty$, whose values are fixed since g_t is given. We describe this by introducing random walks ζ_i, $i\in\mathcal{T}$ on the set $\{1,\ldots,|\mathcal{T}|\}$. $\zeta_i=u$ means that there are $u-1$ particles in $\mathcal{T}$ with position less than x_i. For notational convenience we renormalize times; T_j becomes j and we agree to denote by $\bar{n}$ the time such that $T_{\bar{n}}<\infty$, $T_{\bar{n}+1}=\infty$. Given $s\leqslant\bar{n}$ let $x: M_{x,x+1}(T_s)\neq 0$ and let $l-1$ be the number of particles in $\mathcal{T}$ which have positions smaller than x (which only depends on g_t). Denote by $\delta_{l,l+1}(s)$ or sometimes simply $\delta(s)$, the value of $M_{x,x+1}(T_s)$. When $\delta_{l,l+1}(s)=1$ at that time the ζ particles which are at l and $l+1$ exchange their position; they do not if $\delta_{l,l+1}(s)=-1$. The $\delta(s)$ are independent symmetric random variables; therefore given g_t the process $\zeta_i(s)$, $i\in\mathcal{T}$, $s\leqslant\bar{n}$, is determined and g_t together with a ζ trajectory fully determines a complete trajectory.

We denote by ρ_{g_t} (or simply ρ if no confusion arises) the law of this process, and given g_t we consider $\mathbf{x}(\cdot)$ and $\mathbf{x}^0(\cdot)$ as random variables on such a process.

In the sequel we will use the following notation: $n_{ij}(t)$ is the number of marks appearing within time t referring to particles i and j; for $i > j$, $\chi(z_i, z_j) = +1$ if $z_i \geqslant z_j$, $= -1$ otherwise.

We estimate the right-hand side of Eq. (5.9) by conditioning to g_t. A first estimate is given in the following (see Note 3):

Lemma 5.2. Let $f(\mathbf{z})$ be a bounded function and $D_{ij}(t)$ as in Eq. (4.3). Let $i > j$, $l > m$, $\mathfrak{I}_1 = \{i, j\}$, $\mathfrak{I}_2 = \{1, m\}$, $\mathfrak{I}_1 \cap \mathfrak{I}_2 = \emptyset$; then

$$\mathbf{E}_Q\big(f(\mathbf{x}(t))D_{i,j}(t)^2\big) = \mathbf{E}_Q n_{i,j}(t) f(\mathbf{x}(t)) \tag{5.10}$$

$$\mathbf{E}_Q(f(\mathbf{x}(t))D_{i,j}(t)) = \mathbf{E}_Q 1(\{n_{i,j}(t) \geqslant 1\}) f(\mathbf{x}(t))\chi(x_i(t), x_j(t)) \tag{5.11}$$

$$\begin{aligned}\mathbf{E}_Q(f(\mathbf{x}(t))D_{i,j}(t)D_{l,m}(t)) &= \mathbf{E}_Q 1(\{n_{i,j}(t) \geqslant 1\}) 1(\{n_{l,m}(t) \geqslant 1\}) \\ &\quad \times f(\mathbf{x}(t))\chi(x_i(t), x_j(t))\chi(x_l(t), x_m(t))\end{aligned} \tag{5.12}$$

$$|\mathbf{E}_Q(f(\mathbf{x}(t))D_{i,j}(t)D_{j,l}(t))| \leqslant 2\mathbf{E}_Q|f(\mathbf{x}(t))| \tag{5.13}$$

Proof. The proof is based on the very simple expression the $D_{i,j}(t)$'s have after conditioning to g_t.

Proof of Eq. (5.10). Let $\mathfrak{I} = \{i, j\}$. Define

$$\varphi(s) = \begin{cases} 1 & \text{if } \zeta_i(s-1) > \zeta_j(s-1) \\ -1 & \text{otherwise} \end{cases} \tag{5.14}$$

then

$$D_{i,j}(t) = -\sum_s \delta(s)\varphi(s) \tag{5.15}$$

$$D_{i,j}(t)^2 = n_{i,j}(t) + \sum_{s \neq s'} \delta(s)\varphi(s)\delta(s')\varphi(s') \tag{5.16}$$

$$\begin{aligned}&\mathbf{E}_Q\big(f\big(\mathbf{x}(t)D_{ij}(t)^2\big) \\ &= \int d\bar{P}(g_t)\mathbf{E}_\rho\Big(f(\mathbf{x}(t))\Big[n_{i,j}(t) + \sum_{s \neq s'} \delta(s)\varphi(s)\delta(s')\varphi(s')\Big]\Big) \\ &= \int d\bar{P}(g_t)\Big[n_{i,j}(t)\mathbf{E}_\rho(f(\mathbf{x}(t))) + \sum_{s \neq s'} \mathbf{E}_\rho(f(\mathbf{x}(t))\delta(s)\varphi(s)\delta(s')\varphi(s'))\Big]\end{aligned}$$

For $s' > s$

$$\begin{aligned}&\mathbf{E}_\rho(\delta(s)\varphi(s)\delta(s')\varphi(s')f(\mathbf{x}(t))) \\ &\quad = \mathbf{E}_\rho(\mathbf{E}_\rho(\delta(s)\varphi(s)\delta(s')\varphi(s')f(\mathbf{x}(t))\,|\,\{\delta(u), u \neq s, s'\}\{\delta(s)\delta(s')\})) \\ &\quad = \mathbf{E}_\rho(f(\mathbf{x}(t))\delta(s)\delta(s')\varphi(s)\mathbf{E}_\rho(\varphi(s')\,|\,\{\delta(u), u \neq s, s'\}\{\delta(s)\delta(s')\})) = 0\end{aligned}$$

Proof of Eq. 5.11. We need to estimate $\mathbf{E}_\rho(-\delta(s)\varphi(s)f(\mathbf{x}(t))$: for $s < \bar{s}$, $\bar{s}: t_{\bar{s}+1} > t \geqslant t_{\bar{s}}$ (see Definition 5.1) we condition to $\{\delta(u)u \neq s, \bar{s}\}$ $\{\delta(s)\delta(\bar{s})\}$, and as above the expectation vanishes. Since

$$-\delta(\bar{s})\varphi(\bar{s}) = \chi(x_i(t), x_j(t))$$

Eq. (5.11) is proven.

Proof of Eq. (5.12). We first condition to g_t defined with reference to $\mathcal{T} = (i, j)$. Since $D_{l,m}(t)$ depends only on g_t, we have as before

$$\mathbf{E}_Q(f(\mathbf{x}(t))D_{i,j}(t)D_{l,m}(t)) = \mathbf{E}_Q(f(\mathbf{x}(t))1(\{n_{i,j}(t) \geqslant 1\})\chi(x_i(t), x_j(t))D_{l,m}(t))$$

and from this Eq. (5.12) is obtained by using the same procedure as in the proof of Eq. (5.11).

Proof of Eq. (5.13). $\mathcal{T} = (i, j, l)$. $\varphi(s) = 0$ if $\delta(s)$ does not involve particles i and j; otherwise it is defined as in Eq. (5.14). $\psi(s)$ is defined like $\varphi(s)$ with reference to particles j and l. We need to estimate $\mathbf{E}_\rho(f(\mathbf{x}(t)\delta(s)$ $\varphi(s)\delta(s')\psi(s'))$. Assume $s > s'$. We first consider the trajectories for which (i) $|\varphi(s)| = 1$ and (ii) $\exists v > s: |\varphi(v)| = 1$ and $\varphi(v') = 0$ for $s < v' < v$. This set can be partitioned in pairs of trajectories by fixing $\delta(u)$ for all $u \notin \{s, v\}$ and the product $\delta(s)\delta(v)$. As above the conditional expectation with respect to this partition of $\delta(s)$ vanishes ($\mathbf{x}(t), \delta(s'), \psi(s'), \varphi(s)$ are fixed in each atom). The remaining trajectories are divided into two sets. The first one is defined by the condition that (i) $|\psi(s')| = 1$ and (ii) $\exists v': s > v' > s'$, $|\psi(v')| = 1$, $\psi(v'') = 0$ for $v' > v'' > s'$. This set is also partitioned in pairs by fixing $\delta(u)$ for $u \notin \{s', v'\}$ and the product $\delta(s')\delta(v')$. Just as before the conditional expectation of $f(\mathbf{x}(t))\delta(s)\varphi(s)\delta(s')\psi(s')$ vanishes. Denote by $C(s, s')$ the other trajectories; then we have

$$\begin{aligned}
&\left|\mathbf{E}_\rho\left(f(\mathbf{x}(t))\sum_{s>s'}\varphi(s)\delta(s)\psi(s')\delta(s')\right)\right| \\
&\quad = \left|\mathbf{E}_\rho\left(f(\mathbf{x}(t))\sum_{s>s'}1_{C(s,s')}\varphi(s)\delta(s)\psi(s')\delta(s')\right)\right| \\
&\quad \leqslant \mathbf{E}_\rho\left(|f(\mathbf{x}(t))|\sum_{s>s'}|\varphi(s)\psi(s')|1_{C(s,s')}\right) \leqslant \mathbf{E}_\rho(|f(\mathbf{x}(t))|)
\end{aligned}$$

since $\{|\varphi(s)\psi(s')| = 1\}C(s, s') \cap \{|\varphi(s'')\psi(s''')| = 1\}C(s'', s''') = \emptyset$, *if* (s, s') $\neq (s'', s''')$. This is so because if $s'' > s$ and a trajectory is in $C(s, s')$ and $|\varphi(s)| = 1$, then it must be $\varphi(v) = 0$ for all $v > s$ and in particular $\varphi(s'')$ $= 0$. Therefore we are reduced to consider cases with $s = s''$. Assume $s''' > s'$; then if a trajectory is in $C(s, s')$ and $|\psi(s')| = 1$ it must be $\psi(v') = 0$ for all $s > v' > s'$, and in particular $\psi(s''') = 0$ since $s''' < s'' = s$ and $s''' > s'$ by assumption. ■

We now proceed to the estimate of Eq. (5.9) by use of Lemma 5.2. The term in Eq. (5.11) is treated in the following:

Lemma 4.3. Let f be a bounded function with bounded derivative and denote as before $t = \epsilon^{-2}\tau$; then

$$\lim_{\epsilon\to 0}\Big[\int dQ\, f(\epsilon\mathbf{x}(t))\chi(x_i(t),x_j(t))1(\{n_{i,j}(t)\geqslant 1\}) - \int dP^0\, f(\epsilon\mathbf{x}^0(t))\chi(x_i^0(t),x_j^0(t))\Big] = 0$$

Proof. We can eliminate the condition $n_{i,j}\geqslant 1$ since f is bounded and

$$\int dQ\, 1(\{n_{i,j}(t)=0\}) = \int d\tilde{Q}^\pi 1\big(\{x_i^0(t')\neq x_j^0(t')\ 0\leqslant t'\leqslant t\}\big) = \int dP^0 1\big(\{x_i^0(t')\neq x_j^0(t')\ 0\leqslant t'\leqslant t\}\big)\xrightarrow[t\to\infty]{}0$$

where $\tilde{Q}^\pi$ is defined in R.5, Section 4, with $\pi(i)=1$, $\pi(j)=2$. By definition of $\tilde{Q}^\pi$, see R.6, Section 4, $\chi(x_i(t),x_j(t)) = \chi(x_i^0(t),x_j^0(t))$ if $x_i^0(t)\neq x_j^0(t)$, so we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int d\tilde{Q}^\pi f(\epsilon\mathbf{x}(t))\chi(x_i(t),x_j(t)) &= \lim_{\epsilon\to 0}\int d\tilde{Q}^\pi f(\epsilon\mathbf{x}(t))\chi\big(x_i^0(t),x_j^0(t)\big)\\ &\quad\times 1\big(\{|x_l(t)-x_l^0(t)|<t^\alpha,\ l=1,\ldots,n\}\big)\\ &= \lim_{\epsilon\to 0}\int d\tilde{Q}^\pi f\big(\epsilon\mathbf{x}^0(t)\big)\chi\big(x_i^0(t),x_j^0(t)\big)\\ &\quad\times 1\big(\{|x_l(t)-x_l^0(t)|<t^\alpha,\ l=1,\ldots,n\}\big)\end{aligned}$$

where R.7 with $\frac{1}{4}<\alpha<\frac{1}{2}$ has been used to get the second equality; and the last one follows by noting that

$$f(\epsilon\mathbf{x}(t)) = f\big(\epsilon\mathbf{x}^0(t)\big) + \epsilon\big(\mathbf{x}(t)-\mathbf{x}^0(t)\big)\psi\big(\epsilon\mathbf{x}^0(t)\big)$$

with ψ bounded and

$$|\epsilon\big(x(t)-x^0(t)\big)|\leqslant \epsilon t^\alpha = \epsilon\epsilon^{-2\alpha}\tau^\alpha \quad\blacksquare$$

Lemma 5.4. Let f as in Lemma 5.3 and $t=\epsilon^{-2}\tau$. Define $n_{i,j}^0(t)$ in the free process as $\mathrm{Card}(\{t'\leqslant t : x_i^0(t')=x_j^0(t')\})$; then

$$\lim_{\epsilon\to 0}\Big(\int dQ\,\epsilon n_{i,j}(t)f(\epsilon\mathbf{x}(t)) - \int dP^0\,\epsilon n_{i,j}^0(t)f\big(\epsilon\mathbf{x}^0(t)\big)\Big) = 0$$

Proof. We first notice that

$$\limsup_{\epsilon\to 0}\int dQ\,\epsilon n_{i,j}(\epsilon^{-2}\tau)<\infty, \qquad \limsup_{\epsilon\to 0}\int dQ\,\epsilon^2 n_{ij}^2(\epsilon^{-2}\tau)<\infty \quad (5.17)$$

This can be obtained using the coupling $\tilde{Q}^{\pi}$, $\pi(i)=1$, $\pi(j)=2$ since $n_{i,j}(t) \leqslant 2n^0_{i,j}(t)$ and then the estimate is reduced to the corresponding one for free particles. We will need a more refined estimate than $n_{i,j}(t) \leqslant 2n^0_{i,j}(t)$ in the sequel, to prove the lemma. We have

$$\lim\left(\int dQ\, \epsilon n_{i,j}(t) f(\epsilon \mathbf{x}(t)) - \int d\tilde{Q}^{\pi}\, \epsilon n_{i,j}(t) f\left(\epsilon \mathbf{x}^0(t)\right)\right) = 0 \tag{5.18}$$

Equation (5.18) is obtained by first introducing $1(\{|x_i(t) - x^0_i(t)| < t^{\alpha}\})$, $i=1,\ldots,n$; $\frac{1}{4} < \alpha < \frac{1}{2}$; the error being negligible because of Eqs. (4.6) and (5.17) (Schwarz's inequality has been used). Then, as in the proof of Lemma 5.3, by using Eq. (5.17), $f(\epsilon \mathbf{x}(t))$ can be changed into $f(\epsilon \mathbf{x}^0(t))$.

We can write the integral over $d\tilde{Q}^{\pi}$ as an integral over dP^0 by considering $n_{i,j}(t)$ as a random variable in this space (see R.6 and Section 4). We then introduce the set of trajectories g^0_t, which differ from one another only by a permutation of $x^0_i(\cdot)$ and $x^0_j(\cdot)$; $dP^0(g^0_t)$ denotes the corresponding law; $\rho^0_{g^0_t}$ (or simply ρ^0) is the law of the process conditioned to g^0_t. Given g^0_t the times $t' \leqslant t$ are determined when $x^0_i(t') = x^0_j(t')$ and we denote them as $t_1, \ldots, t_{\bar{s}}$. We remark that $\mathbf{x}^0(t_n)$, $n \leqslant \bar{s}$, is a function constant on g^0_t. Given g^0_t we write $n_{i,j}(t) = n_{i,j}(t_{\bar{s}}) + n_{i,j}(t_{\bar{s}}, t)$, where the last term denotes the number of collisions of the interacting particles after $t_{\bar{s}}$ up to t. Therefore

$$\begin{aligned}
\int d\tilde{Q}^{\pi}\, \epsilon n_{i,j}(t) f\left(\epsilon \mathbf{x}^0(t)\right) &= \int dP^0\left(g^0_t\right) \mathbf{E}_{\rho^0}\left(\epsilon\left(n_{i,j}(t_{\bar{s}}) + n_{i,j}(t_{\bar{s}}, t)\right) f\left(\epsilon \mathbf{x}^0(t)\right)\right) \\
&= \int dP^0\left(g^0_t\right) \mathbf{E}_{\rho^0}\left(\epsilon n_{i,j}(t_{\bar{s}}) \mathbf{E}_{\rho^0}\left(f\left(\epsilon \mathbf{x}^0(t)\right) \mid \mathbf{x}^0(t_{\bar{s}})\right)\right) \\
&\quad + \int dP^0\, \epsilon n_{i,j}(t_{\bar{s}}, t) f\left(\epsilon \mathbf{x}^0(t)\right)
\end{aligned} \tag{5.19}$$

The last term vanishes as $\epsilon \to 0$ because $n_{i,j}(t_{\bar{s}}, t) \leqslant 1$ by definition of $\tilde{Q}^{\pi}$. Then

$$\begin{aligned}
\lim \int d\tilde{Q}^{\pi}\, \epsilon n_{i,j}(t) f\left(\epsilon \mathbf{x}^0(t)\right) - \int dP^0\left(g^0_t\right) \mathbf{E}_{\rho^0}\left(\epsilon n_{i,j}(t_{\bar{s}})\right) \\
\times \mathbf{E}_{\rho^0}\left(f\left(\epsilon \mathbf{x}^0(t)\right) \mid \mathbf{x}^0(t_{\bar{s}})\right) = 0
\end{aligned}$$

because as already remarked $x^0(t_{\bar{s}})$ depends only on g^0_t. It is easy to see, from definition of $\tilde{Q}^{\pi}$ that

$$\mathbf{E}_{\rho}\left(\epsilon n_{i,j}(t_{\bar{s}})\right) = \epsilon \bar{s} = \epsilon n^0_{i,j}(t)$$

Proof of Theorem 3.1. By Theorem 5.2 and Lemma 5.1, the theorem follows from the evaluation of the limits in the right-hand side of Eq. (5.9).

We can rewrite the terms like $\epsilon(z_i - z'_i)$ and $\epsilon^2(z_i - z'_i)(z_j - z'_j)$ using Eq. (4.3), and Lemma 5.2. The terms with $D_{i,j} D_{l,m}$ have a factor ϵ and their

expectation vanishes since f as given in Eq. (5.9) is uniformly bounded. From Eqs. (5.10) and (5.11) it follows that only the terms with $D_{i,j}$ and $D_{i,j}^2$ give a finite contribution in the limit $\epsilon \to 0$.

From Eqs. (5.9), (5.11), and Lemma 5.3 the first term in Eq. (3.2c) is recovered; the second term is given for $i > j$ by the limit as ϵ goes to zero of

$$\int dP^0 \epsilon n_{i,j}^0(t) p\big(\epsilon x_j^0(t)\big) p''\big(\epsilon x_i^0(t)\big) \tag{5.20}$$

We can rewrite Eq. (5.20) in discrete time by looking at the times when particles move. The number of steps, m, is distributed with a Poisson law of intensity 2, denoted by $P(m;t)$, and, given $m, x_1(k), x_2(k)$, $k = 1, \ldots, m$, are independent symmetric random walks with only one particle moving at a time. In this scheme $n_{i,j}^0(t)$ is given by the following:

$$\text{given } m: \quad n_{i,j}^0(t) = \sum_{k=1}^{m} \sum_{u \in \mathbb{Z}} 1\big(\{x_1(k) = u,\, x_2(k) = u\}\big) \tag{5.21}$$

and so

$$\int dP^0 \epsilon n_{i,j}^0(t) f(\epsilon \mathbf{x}(t)) = \sum_m P(m,t) \sum_{k=1}^{m} \sum_{u \in \mathbb{Z}} \epsilon P_{x_1 x_2}^2(u,u;k) \times \sum_{z_1, z_2} P_{u,u}^2(z_1, z_2; m-k) f(\epsilon z_1, \epsilon z_2) \tag{5.22}$$

where $P_{x_1,x_2}^2(x_1', x_2'; n)$ is the probability that after n steps the two particles starting from x_1, x_2 are at x_1', x_2'. We write $t = \epsilon^{-2}\tau$, $m = \epsilon^{-2}\tilde{m}$, $k = \epsilon^{-2}\tilde{k}$, $u = \epsilon^{-1}\tilde{u}$, $z_i = \epsilon^{-1}\tilde{z}_i$, $x_i = \epsilon^{-1}\xi + x_i'$ and then it is easy to see that the right-hand side of (5.22) converges as ϵ goes to zero to the second term of Eq. (3.2c) with $\xi = \eta$. ■

Remark. Equation (2.12) can be easily derived from the previous theorem. In fact, we have to evaluate, $t = \epsilon^{-2}\tau$

$$\begin{aligned} &\epsilon^{-1} \sum_{\mathbf{z}} P\big(\{\mathbf{x}(t) = \mathbf{z}\}\big) \prod_{i=1}^{n} \mu^\epsilon(\eta(z_i) = 1) \\ &\quad = \epsilon^{-1} \Bigg\{ \sum_{\mathbf{z}} \Big[P\big(\{\mathbf{x}(t) = \mathbf{z}\}\big) - P^0\big(\{\mathbf{x}^0(t) = \mathbf{z}\}\big) \Big] \prod_{i=1}^{n} \mu^\epsilon(\eta(z_i) = 1) \\ &\qquad + \prod_{i=1}^{n} \pi(x_i, t; \epsilon) \Bigg\} \end{aligned} \tag{5.23}$$

where $\pi(x,t;\epsilon)$ is defined by Eq. (2.5) and $P[P^0]$ is the law of the interacting (free) process starting on $\mathbf{x} = (x_1, \ldots, x_n)$.

We approximate $\mu^\epsilon(\eta(z_i) = 1)$ with $p(\epsilon z_i)$, and after an expansion as in Lemma 5.1, the terms which give a nonzero contribute to the limit are like in Eqs. (5.10) and (5.11) of Lemma 5.2. They can be estimated as in Lemmas 5.4 and 5.3, respectively. The only difference is that it is not

possible to eliminate the characteristic function 1 ($n_{i,j}(t) \geqslant 1$) as in Lemma 5.3 because in this case particles i and j might start from far apart.

6. NOTES

Note 1. For case of reference we quote here the results concerning the approach to equilibrium and local equilibrium for the symmetric simple exclusion process.[1,2,13] Define for $\eta \in \{0,1\}^{\mathbb{Z}}$, $x \in \mathbb{Z}$, $t \in \mathbb{R}_+$

$$\pi(x,t,\eta) = \frac{1}{(2\pi)t^{1/2}} \sum_{y \in \mathbb{Z}} \eta(y) e^{-(y-x)^2/2t}$$

Then we have the following:

Theorem. $\mu_t : T_t\mu$ converges weakly to ν iff $\pi(0,t,\eta)$ converges μ–a.s. as $t \uparrow \infty$; in such a case let $p(\eta)$ be its limit, then

$$\nu = \int \mu(d\eta)\, \nu_{p(\eta)}$$

(ν_p as usual denotes the Bernoulli measure with parameter p.) μ_t exhibits local equilibrium behavior in the sense of P.1 iff there exists a function $p(x,t \mid \mu)$, $x \in \mathbb{Z}$, $t \in \mathbb{R}_+$ such that for every $\delta > 0$

$$\lim_{t \to \infty} \sup_{x \in \mathbb{Z}} \mu(\{|\pi(x,t,\eta) - p(x,t \mid \mu)| > \delta\}) = 0$$

It is easy to see that if the average $1/(2n+1)\sum_{|x| \leqslant n} \eta(x)$ converges to $p(\eta)$ as n diverges then $\pi(0,t,\eta)$ also converges to $p(\eta)$. A sufficient condition for local equilibrium to hold is the following one: for each $n \in \mathbb{N}$ there is a positive decreasing function $\varphi_n(x)$ $x \in \mathbb{N}$, $\lim_{x \to \infty} \varphi_n(x) = 0$ such that for all $n \in \mathbb{N}$, $(x_1, \ldots, x_n) \in \mathbb{Z}^n$

$$\left| \mu(\{\eta(x_i) = 1, i = 1, \ldots, n\}) - \prod_{i=1}^{n} \mu(\{\eta(x_i)\}) \right| \leqslant \varphi_n\Big(\min_{x_i \neq x_j} |x_i - x_j| \Big)$$

Note 2. The results and proofs for the stationary case are given in Ref. 15. The proof appearing in a preliminary version of this paper[16] was wrong. To fix it we have been forced to use rather different techniques and this is the reason why we present the results in two joint papers.

Note 3. The estimates in Lemma 5.2 can be used for a direct proof that the fluctuation field converges to a Gaussian process (cf. Ref. 3). We can show that the moments of the fluctuation field converge to the values given by the limiting Gaussian field.

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